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Mathematical competencies in higher education: Epistemological beliefs and critical thinking in different strands of pre-service teacher education

Abstract

Assessment in higher education requires multifaceted instruments to capture competency structures and development. We investigate two aspects of competencies of pre-service mathematics teachers: a certain aspect of mathematical abilities (critical thinking with respect to mathematical problem situations) and epistemological beliefs (assessed by both belief position and belief justification). We investigated 463 students from two universities with respect to both aspects of competencies. We show that students' belief position and justification are independent and can be assessed independently. Whereas belief position is not correlated with the number of the students' semesters, their course of studies, and their mathematical abilities, belief justification is indeed correlated with these factors.

Keywords

Epistemological Beliefs; Critical Thinking; Mathematical Competencies

Mathematische Kompetenzen in der Hochschulbildung: Epistemologische Überzeugungen und kritisches Denken in verschiedenen Zweigen der Lehrerausbildung

Zusammenfassung

Leistungsmessung in der Hochschulbildung benötigt facettenreiche Instrumente, um Kompetenzstrukturen und -entwicklung erfassen zu können. Wir untersuchen zwei Aspekte von Kompetenzen von Studierenden des Lehramts Mathematik: einen bestimmten Aspekt mathematischer Fähigkeiten (kritisches Denken mit Bezug

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auf mathematische Problemsituationen) und epistemologische Überzeugungen (wobei sowohl die Position als auch die Begründung der Überzeugungen erfasst werden). Wir haben Daten von 463 Studierenden von zwei Universitäten zu beiden Kompetenzaspekten erhoben und ausgewertet. Wir zeigen, dass die Positionen und Begründungen der studentischen Überzeugungen unabhängig voneinander sind und unabhängig voneinander erhoben werden können. Während die Überzeugungsposition nicht korreliert ist mit der Anzahl der Semester der Studierenden, ihrer Studienwahl und ihren mathematischen Fähigkeiten, zeigen sich bei der Begründung der Überzeugungen tatsächlich Korrelationen mit diesen Faktoren.

Schlagworte

Epistemologische Überzeugungen; kritisches Denken; mathematische Kompetenzen

1. Introduction

Teacher education addresses a wide spectrum of knowledge, skills, and beliefs which are currently regarded as aspects of the broader construct of teacher competency (Shulman, 1986; Krauss et al., 2008; Blömeke, Kaiser, & Lehmann, 2008). When focusing on those competencies that are specific for the teachers' teaching subject, various models of subject matter knowledge and pedagogical content knowledge are discussed (e.g., for mathematics Hill, Shilling, & Ball, 2004; Depaepe, Verschaffel, & Kelchtermans, 2013). The fact that knowledge and beliefs have an impact on teaching quality and learning outcome (Baumert et al., 2010; Voss, Kleickmann, Kunter, & Hachfeld, 2013) is widely acknowledged. However, the evidence on how both aspects are connected and how they develop during teacher education still mainly draws on the evidence from case studies and is in need for the development of systematic measurement approaches (Blömeke, Zlatin-Troitschanskaia, Kuhn, & Fege, 2013). With respect to *mathematical knowledge*, existing instruments mostly strive to measure the amount of knowledge that has been accumulated during university curriculum (Döhrmann, Kaiser, & Blömeke, 2014; Krauss et al., 2008). With respect to mathematical *beliefs*, one can find a broad spectrum of conceptualizations and instruments which produce partly contradictory findings (see section 1 in the theoretical background). Studies that report on development of knowledge and beliefs during teacher education typically draw on cross-sectional data and investigate the two aspects independently (Voss et al., 2013; Kraus et al., 2008; Kleickmann et al., 2013).

In our approach, we wish to contribute to the efforts to measure the development of mathematical beliefs and knowledge during the first years of university study. We are emphasizing two aspects: Firstly, we assess *both* aspects, beliefs and knowledge, simultaneously so that we can answer questions with respect to

the connection between the developments in both areas. Secondly, we assess both aspects in a specific manner: Our focus is not the accumulation of subject matter knowledge, but the growing ability to flexibly solve mathematical problems. Also, we assess mathematical beliefs not by an approach that considers certain beliefs toward mathematics as naïve and others as sophisticated, but we rather differentiate between the belief position (sometimes called *belief orientation*: whether mathematical knowledge is considered to be rather certain or uncertain) and the belief justification (how thoroughly the beliefs are supported by reasons and arguments).

For both purposes we developed instruments that will be described in the following sections. We applied the instruments in several populations of students from different systems of mathematics teacher education. The intention was to test whether the findings can be interpreted consistently with the goals and the structure of the different systems and whether one can find overall structures connecting knowledge and beliefs at different points of teacher education.

Since the aim of the study is not an exhaustive definition of teacher competency or the test of a broad model, as do many large-scale-studies, we concentrate on certain dimensions which we consider central and relevant for a better understanding of the competence structure and development in teacher education.

2. Theoretical Background

2.1 Epistemological beliefs

With respect to the belief aspect, we focus on epistemological beliefs, which are beliefs concerning the nature of knowledge and knowing. Epistemological beliefs are a topic of psychology and (mathematics) education that has received a growing interest among researchers in recent years. The impact of epistemological beliefs on gaining and processing knowledge as well as on teaching and learning in general are widely recognized (Buehl & Alexander, 2006; Hofer & Pintrich, 1997). Furthermore, students' epistemological beliefs are affected by their teachers' epistemological beliefs and their teaching style (e.g., Brownlee & Berthelsen, 2008).

Most researchers agree that there are several dimensions of those beliefs (cf. Hofer & Pintrich, 1997) and that learners can pass several stages of development in each of those dimensions during their school and university education. Such a development is often considered to start with fixed, absolutistic beliefs and can reach flexible, cross-linked, evaluativistic beliefs. Hofer and Pintrich (1997) present an extensive review of developmental models of epistemological beliefs, stating that most researchers agree on a hierarchical sequence of stages that describes the development of such beliefs. However, there is no consensus on such stages or even on the number of stages.

However, the *theoretical* foundation as well as the empirical research on epistemological beliefs are heterogeneous (Bromme, Kienhues, & Stahl, 2008). Recently,

beliefs are recognized as rather context-specific than general which should be taken into account in a more systematic way (Hofer, 2000). Furthermore recent studies suggest that it is not sufficient to solely capture the general position of beliefs (e.g., “mathematical knowledge is certain vs. uncertain”) but also the context in which statements on beliefs arise (Greene & Yu, 2014).

In addition to these theoretical issues, there are *methodological* issues regarding the instruments used by a majority of the studies in both mathematics education and psychology as those instruments mainly consist of closed question formats (Duell & Schommer-Aikins, 2001). In psychological research, the most common method of measuring epistemological beliefs is the use of questionnaires that build on Schommer’s (1990) questionnaire, which uses Likert scale items (cf. Hofer, 2000). In mathematics education, studies which use closed items are also very widespread. For example, the COACTIV study (Baumert et al., 2009, pp. 63 ff.), building on Grigutsch, Raatz, and Törner. (1998), used questions with a four-point Likert scale (is not correct/is rather not correct/is rather correct/is correct). Example items are: “Mathematics is characterized by rigor, namely a rigor in definitions and formal strictness in the mathematical argumentation.” or “Mathematics is a logically consistent thought structure with precisely defined terms and uniquely provable statements.” In this line of research, the expressed beliefs (e.g., “mathematics is a rigorous science”) which we call *belief position* is assumed to be related to the justification of the belief. For example, beliefs on the certainty of knowledge are regarded to be absolutistic and inflexible, whereas beliefs on the uncertainty of knowledge are interpreted as evaluativistic and sophisticated (e.g., Hofer & Pintrich, 1997; Schommer, 1998; Muis, 2004).

Stahl (2011, p. 41 f.) argues that there has been little success in developing a questionnaire with strong reliability and validity. The main problem according to Stahl is the unstable factor structure of the instruments; he identifies another problematic aspect in items which are often indirectly related to epistemological beliefs. Muis (2004) points out additional difficulties with questionnaires in their effectiveness and in their capability of measuring general as well as domain-specific epistemological beliefs.

To overcome these theoretical and methodological issues, in an extension of the previous measuring procedures, we tried a different approach to assess epistemological beliefs. We believe that at least for the domain of mathematics, the belief position is not tied to its justification in such a strong way as the psychological research suggests (see the methodological issues above). For example, a person might hold the position that mathematical knowledge is uncertain and he or she might have more sophisticated arguments (e.g., the possibility of errors in the review and publication process and examples of published but erroneous proofs) or less sophisticated arguments (“Every knowledge is uncertain. I do not believe in absolute truth.”) to back up his or her belief position. This view of justification is in line with Stahl’s (2011) theory on cognitive flexibility and Bromme et al. (2008, p. 432) who describe sophisticated epistemological beliefs “as those beliefs which allow for context-sensitive judgments about knowledge claims.”

In a qualitative interview study (Rott, Leuders, & Stahl, 2014), we could show that both belief positions – *mathematical knowledge is certain* vs. *uncertain* – can be held with sophisticated as well as naïve and inflexible arguments. Therefore, we strived for a quantitative instrument to capture not only individuals’ belief positions but also the ways their belief positions are justified to obtain a more valid picture of peoples’ beliefs. Based on the interview questions, we developed a questionnaire with open-ended questions instead of Likert-scale responses which is a precursor of the instrument described below and investigated mathematics pre-service teachers with respect to their epistemological beliefs on the certainty of mathematical knowledge. (We also investigated their mathematical abilities; see the paragraph on Mathematical critical thinking.) In a questionnaire study with 215 pre-service teachers (Rott, Leuders, & Stahl, 2015), we could show that epistemological beliefs can be measured in two dimensions – belief position and belief justification – and that these dimensions are independent of each other which at least partly contradicts previous research (see above). In our research on mathematics-related epistemological beliefs, these two belief dimensions were not related. Participants judged mathematical knowledge as either certain or uncertain either inflexibly or sophisticatedly. In that specific group, the relative frequency of sophisticated answers was higher for fourth semester students than for first semester students, indicating an increase of reflection on beliefs in this pseudo-longitudinal survey.

2.2 Mathematical critical thinking

As a second aspect of mathematics-related competency, we focus on a dimension that relates to mathematical knowledge and its flexible application. It is not the goal of our study to comprehensively measure mathematical knowledge in teacher education (such as Baumert et al., 2010; Blömeke et al., 2008; Voss et al., 2013). Instead, we want to tap on an aspect of knowledge which reflects the flexibility of students to deal with unknown mathematical situations and which is rather independent of the mere accumulation of content knowledge. We assume that this type of mathematical flexibility is more developed in the same individuals that also argue more flexibly when asked to justify their beliefs. This assumption would be in line with the concept of a *reflective mind* as introduced below.

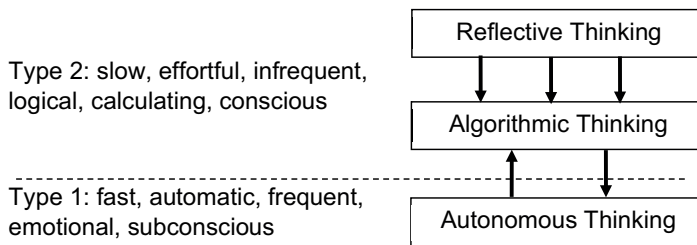
The instrument we developed in order to assess this aspect of mathematical knowledge was inspired by certain arguments from the research on critical thinking. Facione (1990, p. 3) “understand[s] critical thinking to be purposeful, self-regulatory judgment which results in interpretation, analysis, evaluation, and inference, as well as explanation of the evidential, conceptual, methodological, criteriological, or contextual considerations upon which that judgment is based.” Though many different conceptualizations of critical thinking exist (e.g., in philosophy, psychology, and education) the following abilities are commonly agreed upon (cf. Lai, 2011, p. 9): analyzing arguments, claims, or evidence; making inferences

using inductive or deductive reasoning; judging or evaluation and making decisions; or solving problems.

For the purpose of our study it is necessary to determine a narrower focus within this broad construct, and it seems reasonable to refer to a model by Stanovich and Stanovich (2010, p. 210 ff.). They build on *dual process theory* in which cognitive activities are distinguished into a fast, automatic, emotional, subconscious (type 1) and a slow, effortful, logical, conscious (type 2) subset of minds (see Kahneman, 2011, for details). Stanovich and Stanovich locate critical thinking within a tripartite model of thinking, an extension of dual process theory, by further differentiating conscious thinking into *algorithmic* and *reflective thinking*, (see Figure 1). Within this model, they interpret critical thinking as a process of monitoring of problem solving activities: Only with the reflective mind, issues of rationality come into play, assessing the efforts of the algorithmic mind. For instance, with this construct it can be explained why people with equal abilities in algorithmic thinking differ in solving complex tasks (cf. *ibid.*, p. 212 ff.).

Within this model, an indicator of critical thinking is the ability to monitor or evaluate problem solving processes with the reflective mind and to take more complex or less self-evident aspects of a problem into consideration.

Figure 1: The tripartite model of thinking by Stanovich and Stanovich (2010, p. 210); the broken horizontal line represents the key distinction in dual process theory.



A typical situation, which demands critical thinking to override algorithmic mathematical solutions by reflective and evaluative processes, is the paradigmatic bat-and-ball task by Kahneman and Frederick (2002): “A bat and a ball cost \$1.10 in total. The bat costs \$1 more than the ball. How much does the ball cost?” The spontaneous, algorithmically or even autonomously produced answer is \$0.10. People who think critically would question this answer and realize that the ball should cost \$0.05, whereas people who do not use critical thinking do not evaluate their first thoughts. This type of situation is typical for many mathematical problems and will be used as a model task for the construction of an instrument to measure critical mathematical thinking.

As said before, the conceptualizations of mathematical beliefs and knowledge described above are far from exhaustive with respect to mathematical competency.

However, we regard them as sufficiently relevant to reflect central dimensions and to investigate their interconnection and development.

In the aforementioned study with 215 pre-service teachers (Rott et al., 2015), we additionally investigated the pre-service teachers' mathematical abilities (defined as critical thinking with respect to mathematical problem situations). We could show that the ability to think critically and the level of critical thinking does not depend on belief position but goes along with the level of justification of the epistemic judgments expressed by pre-service teachers. Also, this pseudo-longitudinal survey hinted at an increase of the ability to think critically as the fourth semester students outperformed the first semester students significantly.

3. Goals and Research Questions

The preceding study (as described in the previous section) was restricted to only 200 pre-service teachers from one university (Rott et al., 2015). Therefore, these results are considered as preliminary and will be tested with a larger number of pre-service teachers from two universities. By conducting a more comprehensive study we intend to a) replicate the previous results, to b) to increase their validity by applying them to different types of teacher education, and to c) test their usability as instruments for the evaluation of the outcome of tertiary education.

The research questions that guide the previous study and the one presented here are:

- Can the students' flexibility in belief justification be validly distinguished from their belief position?
- Is there a connection to the knowledge domain, more precisely their critical thinking skills operationalized as the flexibility with which students approach mathematical problems?
- Are there differences in the beliefs and the critical thinking skills between students of different numbers of semesters and different educational programs?

These questions are addressed by the following hypotheses that will be tested within the study at hand.

- (H1) *Hypothesis 1*: The two theoretical dimensions of epistemological beliefs – belief position (*certain* vs. *uncertain*) and belief justification (*inflexible* vs. *sophisticated*) – are empirically distinguishable.

This will be tested by means of a belief questionnaire with open-ended items in the same manner as in Rott et al. (2015). The independence of the two dimensions should be true for the whole population as well as for all relevant subgroups (i.e., low or high number of semesters and differing educational programs at the universities).

- (H2) *Hypothesis 2*: For the distribution of judgments regarding the certainty of mathematical knowledge, i.e., the belief positions (*certain* vs. *uncertain*), we

do not expect significant differences between the relevant subgroups (i.e. low or high number of semesters and educational programs at the universities).

However, we do expect significant differences between the relevant subgroups for the distribution of the participants' belief justification (*inflexible* vs. *sophisticated*):

- (H3) *Hypothesis 3*: Students with a higher number of semesters argue in a more sophisticated way than students with a lower number of semesters. We also expect students with more mathematics-related content in their university education (upper secondary teachers) to argue more sophisticatedly than students with less mathematics-related content (primary and lower secondary teachers).

The flexible application of mathematical knowledge had to be measured as independently as possible of the knowledge of certain mathematical concepts learned in specific mathematics courses. To reflect this, we conceptualized mathematical abilities as critical thinking during problem solving. This ability should increase during university education, with a steeper increase in students in strands of teacher education that require more mathematics courses. Regarding the scores in the test on mathematical critical thinking, we anticipate significant differences between the subgroups:

- (H4) *Hypothesis 4*: Students with a higher number of semesters score higher than students with a lower number of semesters. Students with more mathematics-related content in their university education (upper secondary teachers) score higher than students with less mathematics-related content (primary and lower secondary teachers).

The assumed independence of the belief position from the justification with which beliefs are backed-up is reflected in the following hypothesis:

- (H5) *Hypothesis 5*: Regarding the relation of epistemological beliefs and critical thinking, there are no significant differences of critical thinking scores between the relevant subgroups sorted by their belief position (*certain* vs. *uncertain*).

The belief and the knowledge dimensions of our test both reflect certain kinds of sophistication and reflection as described in the tripartite model. Although we do not propose a model of common cognitive processes, we assume that there is a substantial correlation between these dimensions.

- (H6) *Hypothesis 6*: Regarding the relation of epistemological beliefs and critical thinking, there are significant differences of critical thinking scores between the relevant subgroups sorted by their belief justification (*inflexible* vs. *sophisticated*): We expect students that argue sophisticatedly in the beliefs questionnaire to score higher in the critical thinking test than students that argue inflexibly.

4. Methods

4.1 Participants

Becoming a teacher in Germany requires an education at a university and this education depends on the type of school the future teacher aspires to teach at. The education to become a teacher at primary schools (Grundschule, grade 1–4) or lower secondary schools (Hauptschule and Realschule, grade 5–10) usually takes eight to ten semesters and comprises basic level mathematics courses (arithmetic, algebra, geometry). Becoming a teacher at upper secondary schools (Gymnasium, grade 5–13) usually takes ten semesters and requires higher level mathematics courses (analysis, linear algebra) but less educational courses compared to primary and lower secondary schools.

We present the results of a study with $n = 463$ pre-service teachers from two universities (University of Education, Freiburg, $n = 277$ and University of Duisburg-Essen, $n = 186$). The students that aspire to teach at primary schools ($n = 198$) are all enrolled at the University of Education in Freiburg; the students for upper secondary schools ($n = 105$) are all enrolled at the University of Duisburg-Essen; the students for lower secondary schools ($n = 160$) are enrolled either at the university in Freiburg ($n = 79$) or in Essen ($n = 81$). All these students participated voluntarily in this study within the first week of the 2014/15 winter term. They were contacted within lectures and were asked to complete the assessment within the lecture time.

4.2 Instruments

In this study, we identify (a) pre-service mathematics teachers' epistemological beliefs, and (b) their mathematical critical thinking and examine relationships between both constructs. Based on qualitative and quantitative preliminary studies (Rott et al., 2014, 2015), according tests and questionnaires with closed and open items have been developed.

4.2.1 Epistemological beliefs and their degree of justification

In the study at hand, we focus on denotative beliefs, i.e., explicitly stated and reflected beliefs instead of connotative beliefs which are affective and associative (cf. Stahl & Bromme, 2007). To gain access to denotative epistemological beliefs and to the justification of the students' judgments, we constructed a questionnaire based on a preliminary interview study (Rott et al., 2014). The interviewees' answers revealed a broad spectrum of possible responses that helped us to construct according items for a quantitative test. In the resulting questionnaire, we use open-ended questions and prompts (see Table 1) explaining controversial points of view to-

wards mathematics as a scientific discipline to acquire a vivid account of our participants' epistemological judgments and according arguments.

Table 1: Prompts and questions in the questionnaire regarding the certainty of mathematical knowledge

Mathematical Knowledge is Certain	Mathematical Knowledge is Uncertain
<p>"In mathematics, knowledge is valid forever. A theorem is never incorrect. In contrast to all other sciences, knowledge is accumulated in mathematics. [...]"</p> <p>It is impossible, that a theorem that was proven correctly will be wrong from a future point of view. Each theorem is for eternity."</p> <p>(Albrecht Beutelspacher, 2001, p. 235; translated by the first author)</p>	<p>"The issue is [...] whether mathematicians can always be absolutely confident of the truth of certain complex mathematical results [...]."</p> <p>With regard to some very complex issues, truth in mathematics is that for which the vast majority of the community believes it has compelling arguments. And such truth may be fallible.</p> <p>Serious mistakes are relatively rare, of course."</p> <p>(Alan H. Schoenfeld, 1994, p. 58 f.)</p>
<p>a) Which of the two positions regarding the certainty of mathematical knowledge can you identify yourself with?</p> <p>b) Please, give reasons for your judgment regarding the certainty/uncertainty of mathematical knowledge.</p> <p>c) Did you yourself make experiences that support one position or the other?</p> <p>d) Compare the certainty of mathematical knowledge to that of knowledge from other domains. For example, is mathematical more or less certain than knowledge from physics, language sciences or educational sciences?</p>	

We developed a coding manual for the belief position (mathematical knowledge as *certain* vs. *uncertain*) and the level of belief justification (as *inflexible* vs. *sophisticated*). Each student's response to the four questions (a–d) in the questionnaire has been considered as one text and rated individually by two raters. These texts were analyzed with respect to the belief position and the arguments used to back up this position. A text was rated as "sophisticated" when it included arguments that reflected epistemological aspects related to the certainty of mathematical knowledge (e.g., referring to mathematical axioms, rigorous proofs, or peer-review) instead of just referring to personal opinions or knockout arguments (without an explanation for the validity of the argument). The length of an answer was not a criterion for this decision. See Table 2 for examples of the four possible outcomes of the rating of students' responses.

Table 3 shows the resulting interrater reliability; minor discrepancies occurred only when students wrote very short answers. Overall, the reliability score shows very high agreement (Cohen's $\kappa = 0.865$). After calculating the interrater reliability, the differing codes have been rated consensually by the two raters. The consensual ratings can be seen in Table 5.

Table 2: Examples for the students' responses in the open-ended belief questionnaire

	Inflexible	Sophisticated
Certain	"Mathematical knowledge is certain. Mathematical objects like numbers will never change." (SBLP-25)	"Mathematical knowledge is certain, because it is not based on observations and on theses based on these observations. Instead, mathematical knowledge is based on conventions (axioms) and resulting theorems. There may be differences regarding these conventions but not the conclusions." (ESEB-16)
Uncertain	"Mathematical knowledge is uncertain, because arguments can be rebutted." (LAAI-21)	"Mathematical knowledge is uncertain. A proof is only valid, because a majority of humans considers it and the according arguments as valid. Without this approval, it would not be valid anymore. Therefore, the certainty of a proof depends on the judgment of humans and this is not safe." (BJAT-16)

Table 3: Calculation of the interrater reliability coding "Certainty of Mathematical Knowledge"

		Rater 1				
		Certain & inflexible	Certain & sophisticated	Uncertain & inflexible	Uncertain & sophisticated	Sum
Rater 2	Certain & inflexible	203	0	25	1	229
	Certain & sophisticated	0	17	0	0	17
	Uncertain & inflexible	0	0	192	5	197
	Uncertain & sophisticated	0	0	5	15	20
	Sum	203	17	222	21	463

$P_{obs} = 0.922$, $P_{exp} = 0.424$, Cohen's $\kappa = 0.865$

4.2.2 Measuring mathematical critical thinking

In order to measure mathematical critical thinking as described above, more than 20 items similar to and including the bat-and-ball task have been constructed or adapted. A second example is the following task: "In a gamble, a regular six-sided die with four green faces and two red faces is rolled 20 times. You win €25 if a certain sequence of results is shown. Which sequence would you bet on? Choose one: (a) RGRRR (b) GRGRRR (c) GRRRRR." Most students choose option (b) because it contains more instances of G than the other options. Those students do not realize that option (a) is entirely included in option (b) and, therefore, more likely.

All items were related to mathematical situations and demanded only knowledge from lower secondary education. In four preliminary studies – of which three

were quantitative studies and one was a qualitative study using task-based interviews – (Rott & Leuders, 2016), it was investigated whether the items measure computational skills or the willingness to engage in a critical reflection of apparently obvious solutions. Items that did not trigger critical thinking according to the model by Stanovich and Stanovich were discarded. After validation and because we wanted to restrict the time for this test to 20 minutes, the final test consisted of 11 items (for further details see Rott et al., 2015).

All items were rated dichotomously and we used a Rasch model to transform our participants' test scores into values on a one-dimensional competency scale (software RUMM 2030 by Andrich, Sheridan, & Luo, 2009). After eliminating two items because of underdiscrimination (fit residual > 2.5) in connection with floor and ceiling effects, respectively, for each item the model showed good fit residuals (all values between -2.5 and 2.5) and no significant differences between the observed overall performance of each trait group and its expected performance (overall- $\chi^2 = 36.2$; $df = 27$; $p = 0.11$).

5. Results

Firstly, we present the distribution of the pre-service teachers' responses regarding the questionnaire on epistemological beliefs. Secondly, we evaluate our participants' scores on the test of mathematical critical thinking. Thirdly, we investigate possible relations between both aspects of mathematics-related research competency.

5.1 Denotative beliefs: Position and justification

The distribution of students that filled out the belief questionnaire is presented in Table 4. The data is sorted by the four possible outcomes of the two belief dimensions (belief position combined with the degree of justification). Additionally, the students' distribution has been sorted by their number of semesters (into novice students with three or less semesters and into advanced students with four or more semesters respectively, this also happens to be a median split) as well as by their aspired teaching profession (primary, lower secondary, or upper secondary school). The last row shows the total number of students in each of the four belief categories. The low percentage of students arguing sophisticatedly (8.2 %) is in line with the respective percentage of students (12.9 %) in the preliminary quantitative study (Rott et al., 2015).

Table 4: Distribution of the students' denotative epistemological beliefs and their degree of justification

	Certain & inflexible	Certain & sophisticated	Uncertain & inflexible	Uncertain & sophisticated	Sum
Semester ≤ 3	117 (47.4%)	6 (2.4%)	114 (46.2%)	10 (4.0%)	247 (100%)
Semester ≥ 4	86 (39.8%)	11 (5.1%)	108 (50.0%)	11 (5.1%)	216 (100%)
Primary	84 (42.4%)	5 (2.5%)	97 (49.0%)	12 (6.1%)	198 (100%)
Lower Secondary	67 (41.9%)	4 (2.5%)	85 (53.1%)	4 (2.5%)	160 (100%)
Upper Secondary	52 (49.5%)	8 (7.6%)	40 (38.1%)	5 (4.8%)	105 (100%)
All students combined	203 (43.8%)	17 (3.7%)	222 (47.9%)	21 (4.5%)	463 (100%)

Chi-square tests were used to address the question whether the two theoretically claimed dimensions (belief position and according justification, cf. H1) are independent. Table 5 presents the set-up of the data for the chi-square test for all students which are the same numbers as in the last row of Table 4 (Yates chi-square test, corrected for continuity: $\chi^2 = 0.04$, $df = 1$, $p = 0.842$).

Table 5: Comparison of both dimensions of the belief questionnaire for all students; the numbers in brackets indicate expected frequency under the assumption of statistical independence

	Inflexible	Sophisticated	Sum
Certain	203 (201.9)	17 (18.1)	220
Uncertain	222 (223.1)	21 (19.9)	243
Sum	425	38	463
	$\chi^2 = 0.04$	$df = 1$	$p = 0.842$

The chi-square test has also been repeated for all sub-groups to check for possible group specific dependencies (Semester ≤ 3 : $\chi^2 = 0.58$, $p = 0.446$; Semester ≥ 4 : $\chi^2 = 0.08$, $p = 0.777$; Primary schools: $\chi^2 = 1.19$, $p = 0.275$; Lower secondary schools: $\chi^2 = 0.001$, $p = 0.975$ (numbers too small for exact results); Upper secondary schools: $\chi^2 = 0.002$, $p = 0.964$; all $df = 1$). These results confirm our hypothesis 1 that both dimensions are statistically independent. Therefore, in Table 6 we sort the numbers of students by those two dimensions separately.

Table 6: Distribution of the students' denotative epistemological beliefs and their degree of justification

	Certain	Uncertain	Inflexible	Sophisticated	Total
Semester ≤ 3	123 (49.8%)	124 (50.2%)	231 (93.5%)	16 (6.5%)	247 (100%)
Semester ≥ 4	97 (44.9%)	119 (55.1%)	194 (89.8%)	22 (10.2%)	216 (100%)
Primary	89 (44.9%)	109 (55.1%)	181 (91.4%)	17 (8.6%)	198 (100%)
Lower secondary	71 (44.4%)	89 (55.6%)	152 (95.0%)	8 (5.0%)	160 (100%)
Upper secondary	60 (57.1%)	45 (42.9%)	92 (87.6%)	13 (12.4%)	105 (100%)
All students combined	220 (47.5%)	243 (52.5%)	425 (91.8%)	38 (8.2%)	463 (100%)

A *first question* regarding these data concerns the pre-service teachers' judgments: Do they regard mathematical knowledge as certain or uncertain and does this change with the number of semesters (cf. H2)? The ratio of pre-service teachers judging "certain" in semester 3 or less is 123 : 124 (49.8 % certain) compared to 97 : 119 (44.9 %) in semester 4 or higher; there is no significant deviation from the null-hypothesis ("no effect") (Yates chi-square test, corrected for continuity: $\chi^2 = 0.92$, $df = 1$, $p = 0.338$).

This question can be repeated for each sub-group. For pre-service teachers of primary schools, there is no significant difference in the judgment of mathematical knowledge as "certain" in semester 3 or less (42 : 43, 49.4 % certain) compared to students in semester 4 or higher (47 : 66, 41.6 % certain) ($\chi^2 = 0.90$, $df = 1$, $p = 0.343$). There is also no significant difference for pre-service teachers of lower secondary schools in semester 3 or less (51 : 69, 42.5 % certain) compared to students in semester 4 or higher (20 : 20, 50.0 % certain) ($\chi^2 = 0.41$, $df = 1$, $p = 0.522$). There is, however, a significant difference for pre-service teachers of upper secondary schools in semester 3 or less (30 : 12, 71.4 % certain) compared to students in semester 4 or higher (30 : 33, 47.6 % certain) ($\chi^2 = 4.90$, $df = 1$, $p = 0.027$). Except for the pre-service teachers that aspire to teach at upper secondary schools, there are no differences within the subgroups which is in accordance with hypothesis 2.

A *second question* concerns the participants' justification: Do they argue inflexibly or sophisticatedly and does the belief justification differ between the relevant subgroups (number of semesters and educational program, cf. H3)? The number of sophisticated reasoning is higher for students with a higher number of semesters than for students with a lower number of semesters, which is 194 : 22 (inflexible : sophisticated, i.e. 10.2 % sophisticated) (semester ≥ 4) in comparison to 231 : 16 (6.5 %) (semester ≤ 3).

A closer inspection of the sub-groups shows that this effect is not visible in primary students: A larger percentage of students of semester 3 or less (76 : 9, 10.6 % sophisticated) argues sophisticatedly compared to students of semester 4 or higher (105 : 8, 7.1 % sophisticated). The effect is visible, though, for the other two sub-groups: Pre-service teachers of lower secondary schools in semester 3 or less

(117 : 3, 2.5 % sophisticated) show less sophisticated statements compared to students in semester 4 or higher (35 : 5, 12.5 % sophisticated). Pre-service teachers of upper secondary schools in semester 3 or less (38 : 4, 9.5 % sophisticated) also show less sophisticated belief justifications compared to students in semester 4 or higher (54 : 9, 14.3 % sophisticated). Except for the primary school pre-service teachers, the slow shift towards more sophisticated belief justification is in accordance with hypothesis 3.

A closer look at the students in the different educational programs confirms the hypothesis, that a higher percentage of students that aspire to become secondary teachers argue sophisticatedly than students for lower secondary and primary schools (12.5 % compared to 5.0 % and 8.6 %, respectively).

5.2 Critical thinking

The Rasch model of the students' critical thinking ability provides metrical latent variables ranging from -2.83 to 2.81 with low values indicating a low ability. Table 7 presents the means of these values that have been sorted by aspired type of school and by the number of semesters (cf. Rott et al., 2015).

A *t*-test was used to compare the two groups that aspire to teach at lower secondary schools (Freiburg: -0.222 (0.864), $n = 79$; Essen: -0.255 (1.047), $n = 81$). The test showed no significant differences between these two groups ($t = 0.22$, $df = 158$, $p_{\text{two-tailed}} = 0.826$), so that we also combined these two groups in our analyses when necessary.

Table 7: Means (and standard deviations) of Mathematical Critical Thinking, sorted by aspired teaching position (crossing universities) and the number of semesters

	Semester ≤ 3	Semester ≥ 4	Total
Primary	0.031 (0.872) $n = 85$	-0.369 (0.915) $n = 113$	-0.197 (0.916) $n = 198$
Lower secondary	-0.235 (0.868) $n = 120$	-0.252 (1.202) $n = 40$	-0.239 (0.958) $n = 160$
Upper secondary	-0.117 (0.907) $n = 42$	0.354 (0.995) $n = 63$	0.166 (0.984) $n = 105$
Total	-0.123 (0.881) $n = 247$	-0.136 (1.041) $n = 216$	-0.129 (0.958) $n = 463$

A question that can be addressed to these data is whether critical thinking scores are dependent on the semester and the aspired teaching position (cf. H4). Surprisingly, there is no significant advantage in the critical thinking scores for students with a higher number of semesters which could be assumed (and was visible

in the preliminary study). Actually, the three study programs show very different results regarding the development of critical thinking within this pseudo-longitudinal survey: Only in the group of students aspiring to teach at upper secondary schools, the more experienced students show significantly higher critical thinking scores (t -test: $t = 2.46$, $df = 103$, $p_{2-sided} = 0,016$; $d = 0.48$). For lower secondary schools, there is no significant difference ($t = 0.1$, $df = 158$, $p_{2-sided} = 0,921$), and for primary schools, there is even a decline ($t = 3.1$, $df = 196$, $p_{2-sided} = 0,002$; $d = 0.44$). Therefore, it is not recommendable to interpret the results of a two-way ANOVA that has been used to investigate whether there are differences between students of semester three or less compared to students of semester four or greater, and whether there are differences between students of the different study programs.

The part of hypothesis 4 regarding higher critical thinking scores for students with a higher number of semesters can therefore not be confirmed. The critical thinking scores regarding the aspired teaching position, however, indicate that students show higher scores in the mathematical critical thinking test, if they have the more profound mathematics education (upper secondary vs. lower secondary and primary education).

5.3 The relation of epistemological beliefs and critical thinking

Table 8 presents mean values of critical thinking scores in relation to the students' answers on the epistemological belief questionnaire; the numbers of students are identical to those in Table 6.

Table 8: Means (and standard deviations) of Mathematical Critical Thinking, sorted by the dimensions of the belief questionnaire

	Certain	Uncertain	Inflexible	Sophisticated	Total
Semester ≤ 3	-0.125 (0.866)	-0.121 (0.898)	-0.163 (0.858)	0.451 (1.029)	-0.123 (0.881)
	$n = 123$	$n = 124$	$n = 231$	$n = 16$	$n = 247$
Semester ≥ 4	-0.076 (1.089)	-0.185 (1.003)	-0.195 (1.026)	0.384 (1.052)	-0.136 (1.041)
	$n = 97$	$n = 119$	$n = 194$	$n = 22$	$n = 216$
Primary	-0.129 (0.921)	-0.253 (0.913)	-0.239 (0.889)	0.245 (1.103)	-0.197 (0.916)
	$n = 89$	$n = 109$	$n = 181$	$n = 17$	$n = 198$
Lower secondary	-0.397 (1.036)	-0.113 (0.877)	-0.271 (0.936)	0.370 (1.235)	-0.239 (0.958)
	$n = 71$	$n = 89$	$n = 152$	$n = 8$	$n = 160$
Upper secondary	0.282 (0.832)	0.011 (1.148)	0.096 (0.991)	0.656 (0.803)	0.166 (0.984)
	$n = 60$	$n = 45$	$n = 92$	$n = 13$	$n = 105$
All students combined	-0.104 (0.969)	-0.153 (0.949)	-0.178 (0.938)	0.412 (1.029)	-0.129 (0.958)
	$n = 220$	$n = 243$	$n = 425$	$n = 38$	$n = 463$

A *first question* that can be addressed to these data is whether critical thinking scores depend on the belief position (“certain” vs. “uncertain”) (cf. H5). A two-way ANOVA has been used to answer this question: There are no significant differences between students regarding mathematical knowledge as “certain” compared to students regarding it as “uncertain” ($F = .159$; $p = .690$). There are significant differences between students of the different study programs (see above) ($F = 6.200$; $p = .002$); a Tukey Post-Hoc reveals that the differences between the study programs are significant between primary and upper secondary ($p = .002$) as well as between lower and upper secondary ($p = .004$), but not between primary and lower secondary ($p = .907$) education. Also, there is a significant interaction effect ($F = 3.263$; $p = .039$). The results regarding the study programs and the interaction effect have to be interpreted carefully because of the disordinal interaction between study program and number of semesters (see above). Nonetheless, these results confirm our hypothesis 5 that critical thinking scores are not related to the belief position (*certain vs. uncertain*).

The *second question* in this context addresses the correlation of critical thinking scores with the degree of belief justification (cf. H6). Another ANOVA shows that students that argue sophisticatedly score significantly better than students that argue in an inflexible way ($F = 11.386$; $p = .001$; $d = 0.62$; observed statistical power of this effect is 96.1 %). A closer look at the data shows that the significant differences for the belief justification are true for all three subgroups of aspired teaching positions (see also Table 7). Thus, the results confirm our hypothesis 6 that there is a marked connection between justification of beliefs and the ability of thinking critically when solving tasks.

6. Discussion

6.1 Discussion of the hypotheses

In the study at hand, most of the hypotheses that were drawn from the preliminary study (Rott et al., 2015) could be confirmed.

H1, the assumed independence of belief position and justification, has been *confirmed* for the whole test population as well as for each sub-group (pre-service teachers with a lower and higher number of semesters as well as for each of the three aspired school types). This result is particularly important for research on epistemological beliefs since within closed questionnaire surveys the belief position is often used to determine its level of justification. For example, Hofer and Pintrich (1997, p. 119 f.) assume that the belief “absolute truth exists with certainty” is valid only for “lower levels” of belief justification (i.e., it is less sophisticated). Our study shows that the belief position of certainty can be held and supported also with sophisticated arguments – at least in the area under investigation. Further areas remain to be investigated to corroborate the assumption of independence.

H2 stated that there are no significant differences regarding belief position within the sub-groups (low or high number of semesters and educational programs at the universities). This hypothesis was *confirmed* except for the group of upper secondary pre-service teachers. We can only speculate on possible reasons of the change within this group. It might be due to the strong mathematics-related content and experiences of flawed proofs of these students' education that favors a shift to uncertainty beliefs. However, we conducted no genuine longitudinal survey and are unable to capture actual changes in students' beliefs.

H3 predicted differences within the sub-groups regarding the belief justification. This hypothesis was *confirmed* except for the group of primary pre-service teachers. Older students should be better able to argue sophisticatedly against a richer backdrop of knowledge and experience, which has become evident in the data. Also, students with a richer mathematical background (trainees for upper secondary schools) showed a higher percentage of sophisticated justification. We do not know, however, what happened in the sample of pre-service teachers for primary schools.

H4, the assumed increase in critical thinking scores between the students from lower and higher semesters could *not* be *confirmed*; but it was *confirmed* for the different educational programs. The unexpected stagnation with respect to critical thinking in some of the groups may be investigated further with respect to possible group substructures. A reason for this could be anything from a bad day that influenced test results for that specific day to lectures that sustainably affected the willingness to think critically of some participants. However the present data does not allow for such a further investigation. One may argue that the tasks used in the critical thinking test do not require mathematics that are taught at the university level and should therefore not depend on the number of semesters.

H5 and H6, predicting no relations between belief position and mathematical competency (operationalized by our critical thinking test) but significant relations between belief justification and mathematical competency could both be *confirmed*. Picking up the discussion from H1, this result seems to be of high importance. Studies trying to show connections between (epistemological) beliefs and other factors of (mathematical) competencies would be well advised to collect data regarding belief justification not solely with instruments that operationalize it via belief positions. H5 and H6 also allow for hypotheses regarding the development of students' competencies during their university education for subsequent studies.

6.2 Limitations

Due to our specific procedure the study has several limitations.

Firstly, some considerations with respect to the validity of the testing procedures seem to be necessary: Within our study we assumed the perspective of distinguishing between beliefs in a dichotomous manner. This is partly inherent in the design (participants had to decide between two opposing statements) and partly

due to the evaluation methods (dichotomization of the multivariate data). Studies with experts (e.g., Mura, 1993) indicate that this approach becomes invalid, when the individuals have broad experience in their subject and hold complex and differentiated views. For people that have reached a very high degree of justification, such as professional mathematicians, the method of forced choice between two belief positions (e.g., “certain” vs “uncertain”) may lead to invalidities, since such people tend to answer that *both* positions can be adequate depending on the context (Stahl, 2011; Gowers, 2013). The interviews that were conducted in our preliminary qualitative study (Rott et al., 2014) confirm these considerations: Highly sophisticated interviewees refuse to commit themselves to one belief position. However, in those interviews (ibid.) it was found that for students such a dialectic position is not yet in reach and that deciding for one or the other belief position is not triggering irritations. Therefore, we consider the methodological decision to rate the students’ belief positions dichotomously as reasonable for the study at hand.

As stated above, all differences regarding the number of the pre-service teachers’ semester have to be interpreted with care as we did not conduct a longitudinal survey. Within this study, we cannot trace the development of students’ epistemological beliefs in the course of their university studies. Furthermore, we cannot tell whether results in favor of students with more semesters are due to a gain in knowledge or to selection effects, i.e. low-performing students leaving the university. To answer according questions, follow-up studies have to be conducted. For a validation it would be most desirable to use the instrument for investigating change of belief and critical thinking during specific learning environments.

Also the strands may be confounded with the location. Such considerations can be dealt with when we extend our investigations to more than only two universities.

6.3 Theoretical considerations with respect to epistemological beliefs and critical thinking

We do not offer a model that explains the justification of beliefs and the level of critical thinking. Possibly we deal here with two quite different constructs (one more verbal, the other more mathematical) that grow simultaneously. However, Kuhn (1999, p. 22 f.) discusses possible relationships between the development of epistemological beliefs and critical thinking. She argues that the transition from absolutistic to evaluativistic beliefs of the knowledge structure and the ability to think critically are closely related. On the one hand, critical reflections lead to questioning beliefs and to the insight that even experts disagree about important issues. These are important steps in developing more sophisticated epistemological beliefs. On the other hand, an absolutist epistemological understanding favors easy and more direct answers on questions of truth or falsity. Kuhn concludes that individuals who confine themselves to an absolutist epistemology have a low demand

for critical thinking skills and, hence, the impetus to exercise and further develop these skills is slight.

The study presented here, is conducted in mathematics teacher education. However, we assume that the constructs presented here can also be found with respect to other subjects, so that it would be interesting to ask whether similar findings would be encountered, for instance, in the education of science or history teachers. Still, the constructs would partly need different operationalizations (e.g., for critical thinking) or a different emphasis on epistemological aspects of the subject than those relevant for mathematics.

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